# To Increase or Diminish the roots by a Given Quantity:→

To effect this transformation we change the variable in the polynomial $f(x)$ by the substitution $x = y + h$; the resulting equation in $y$ will have roots each less than or greater by $h$ than the given equation in $x$, according as $h$ is positive or negative.

$$f(y+h) = f(h) + f'(h)\cdot y + \frac{f''(h)}{1\cdot 2} y^2 + \frac{f'''(h)}{1\cdot 2\cdot 3} y^3 + \cdots = 0 \quad \text{—①}$$

Let the proposed equation be

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots\cdots + a_{n-1}x + a_n = 0; \quad \text{—②}$$

and suppose the transformed polynomial in $y$ to be

$$A_0y^n + A_1y^{n-1} + A_2y^{n-2} + \cdots\cdots + A_{n-1}y + A_n = 0 \quad \text{—③}$$

$\because y = x - h$, this is equivalent to

$$A_0(x-h)^n + A_1(x-h)^{n-1} + A_2(x-h)^{n-2} + \cdots + A_{n-2}(x-h) + A_n = 0 \quad \text{—④}$$

Which must be identical with the given polynomial

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We conclude that if the given polynomial be divided by $(x-h)$, the remainder $A_n$, & quotient

$$A_0(x-h)^{n-1} + A_1(x-h)^{n-2} + \cdots + A_{n-2}(x-h) + A_{n-1};$$

If this again divided by $x-h$, the remainder is $A_{n-1}$; & the quotient

$$A_0(x-h)^{n-2} + A_1(x-h)^{n-3} + \cdots + A_{n-2}.$$

Proceeding in this way, we are able by a repetition of arithmetical operations, of the kind explained.

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The last $\underline{A_0}$ being equal $\underline{\text{to } a_0}$.

(Q) Find the equation whose roots are the roots of the equation

$$x^4 - 5x^3 + 7x^2 - 17x + 11 = 0$$

each diminished by $\underline{4}$.

$x = y + 4$ $\rightarrow$ $y = x - 4$

$\rightarrow$ The calculation is best exhibited as follows:-

| | | | | | |
|---|---|---|---|---|---|
| 4 | 1 | $-5$ | 7 | $-17$ | 11 |
| | | 4 | $-4$ | 12 | $-20$ |
| | 1 | $-1$ | 3 | $-5$ | $-9 = A_4$ |
| | | 4 | 12 | 60 | |
| | 1 | 3 | 15 | $55 = A_3$ | |
| | | 4 | 28 | | |
| | 1 | 7 | $43 = A_2$ | | |
| | | 4 | | | |
| | $1 = A_0$ | $11 = A_1$ | | | |

Here the required transformed equation is

$$A_0 y^4 + A_1 y^3 + A_2 y^2 + A_1 y + A_4 = 0$$

$$\Rightarrow \; y^4 + 11y^3 + 43y^2 + 55y - 9 = 0.$$

## Removal of terms

$\cancel{a_0 + a_1}$

$$a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1} x + a_n = 0$$

Let $\cancel{y}$ $x = y + h$.

$$\Rightarrow a_0 (y+h)^n + a_1 (y+h)^{n-1} + a_2 (y+h)^{n-2} + \cdots$$

$$\cdots + a_{n-1}(y+h) + a_n = 0$$

$$\Rightarrow a_0 \left\{ y^n + n h y^{n-1} + \frac{n(n-1)}{1\cdot 2} h^2 y^{n-2} + \cdots \right\}$$

$$+ a_1 \left\{ y^{n-1} + (n-1) h y^{n-2} + \cdots \right\} + \cdots$$

$$\cdots$$

$$+ a_{n-1} y + (a_{n-1} h + a_n) = 0$$

$\Rightarrow$ $\cancel{a_0 + a_0}$

$$\Rightarrow a_0 y^n + \underline{(a_0 n h + a_1)} y^{n-1} + \underline{\left\{ \frac{n(n-1)}{1\cdot 2} a_0 h^2 + (n-1) a_1 h \right.}$$

$$\left. + a_2 \right\} y^{n-2} + \cdots = 0.$$

$$\boxed{a_0 n h + a_1 = 0 \Rightarrow h = -\frac{a_1}{a_0 n}}$$

$x - 2 = y \rightarrow x = y + 2$

(8) Transform the equation

$$x^3 - 6x^2 + 4x - 7 = 0$$

onto one which shall want the second term zero.

$\longrightarrow$ $n = \cancel{4}\,3,\ a_0 = 1,\ a_1 = -6$

$\because a_0 n h + a_1 = 0$

$\Rightarrow 1 \cdot \cancel{4}\,3 \cdot h - 6 = 0 \Rightarrow \underline{h = 2}$.

Diminish the roots by $\underline{2}$.

| | | | | |
|---|---|---|---|---|
| 2 | 1 | $-6$ | 4 | $-7$ |
| | | 2 | $-8$ | $-8$ |
| | 1 | $-4$ | $-4$ | $-15 = A_3$ |
| | | 2 | $-4$ | |
| | 1 | $-2$ | $-8 = A_2$ | |
| | | 2 | | |
| | $\underline{1} = A_0$ | $0 = A_1$ | | |

$\therefore$ Transformed equation is $y^3 - 8y - 15 = 0$. $\underline{\text{Ans.}}$

Ex:→ Remove the second term from the equation $a_0x^3 + 3a_1x^2 + 3a_2x + a_3 = 0$ — (1)

solution:→ $n = 3$

$a_0 = a_0$

$h = ?$

$a_1 = 3a_1$

$\therefore\ na_0h + a_1 = 0 \Rightarrow 3a_0h + 3a_1 = 0$

$\Rightarrow h = -\frac{a_1}{a_0}$

So in order to remove the second term we should diminish the root of be given equation (1) by $-\frac{a_1}{a_0}$ i,e, we should increase the root by $\frac{a_1}{a_0}$.

| | | | | |
|---|---|---|---|---|
| $-\frac{a_1}{a_0}$ | $a_0$ | $3a_1$ | $3a_2$ | $a_3$ |
| | | $-a_1$ | $-\frac{2a_1^2}{a_0}$ | $-\frac{3a_2a_1}{a_0} + \frac{2a_1^3}{a_0^2}$ |
| | $a_0$ | $2a_1$ | $3a_2 - \frac{2a_1^2}{a_0}$ | $\frac{a_3a_0^2 - 3a_0a_1a_2 + 2a_1^3}{a_0^2}$ |
| | | $-a_1$ | $-\frac{a_1^2}{a_0}$ | |
| | $a_0$ | $a_1$ | $\frac{3(a_2a_0 - a_1^2)}{a_0}$ | |
| | | $-a_1$ | | |
| | $a_0$ | 0 | | |

so the transformed equation is

$$a_0 y^3 + \frac{3(a_2 a_0 - a_1^2)}{a_0} y + \frac{a_3 a_0^2 - 3a_0 a_1 a_2 + 2a_1^3}{a_0^2} = 0 \quad \text{--- (2)}$$

Here we write

$$G = a_0^2 a_3 - 3a_0 a_1 a_2 + 2a_1^3 \quad \text{--- (3)}$$

& $$H = a_0 a_2 - a_1^2 \quad \text{--- (4)}$$

Then (2) becomes

$$y^3 + \frac{3H}{a_0^2} y + \frac{G}{a_0^3} = 0 \quad \text{--- (5)}$$

If the roots of this equation (5) be multiplied by $a_0$ then it becomes

$$Z^3 + 3HZ + G = 0 \quad \text{--- (6)}$$

So, orginal equation multiplied by $eq^n$ (2) is identical with

$$(a_0 x + a_1)^3 + 3H(a_0 x + a_1) + G = 0. \quad \text{--- (7)}$$

~~If $\alpha, \beta$ and $\gamma$~~
If roots of orginal cubic equation are $\alpha$, $\beta$ and $\gamma$ then the roots of (5) ~~is~~ will be the transformed equation

$$\alpha + \frac{a_1}{a_0}, \quad \beta + \frac{a_1}{a_0}, \quad \gamma + \frac{a_1}{a_0} \quad \text{---(ix)}$$

$$\because \alpha + \beta + \gamma = -\frac{3a_1}{a_0}$$

$$\therefore \frac{a_1}{a_0} = -\frac{1}{3}(\alpha + \beta + \gamma)$$

So the roots in this equation are

$$\alpha - \frac{1}{3}(\alpha+\beta+\gamma), \quad \beta - \frac{1}{3}(\alpha+\beta+\gamma) \ \& \ \gamma - \frac{1}{3}(\alpha+\beta+\gamma)$$

$$\text{or,} \ \frac{1}{3}(2\alpha - \beta - \gamma), \quad \frac{1}{3}(2\beta - \alpha - \gamma) \ \& \ \frac{1}{3}(2\gamma - \alpha - \beta)$$

Sum of these roots $= 0$

$\therefore$ From (5) sum of product of roots taken two by two is

$$\frac{1}{9}\sum(2\alpha - \beta - \gamma)(2\beta - \alpha - \gamma) = \frac{3H}{a_0^2}$$

$$\text{or,} \ \sum(2\alpha - \beta - \gamma)(2\beta - \gamma - \alpha) = \frac{27H}{a_0^2}$$

$$\therefore \sum(2\alpha-\beta-\gamma)(2\beta-\gamma-\alpha) = \frac{27(a_0a_2-a_1^2)}{a_0^2}.$$ Ans.

and product of the roots from (5)

$$\frac{1}{27}(2\alpha-\beta-\gamma)(2\beta-\alpha-\gamma)(2\gamma-\alpha-\beta) = -\frac{G}{a_0^3}$$

$$\Rightarrow (2\alpha-\beta-\gamma)(2\beta-\alpha-\gamma)(2\gamma-\alpha-\beta) = -\frac{27G}{a_0^3}$$

$$= \frac{-27(a_0^2a_3-3a_0a_1a_2+2a_1^3)}{a_0^3}.$$

Remark: →

$$H = a_0a_2 - a_1^2 = \begin{vmatrix} a_0 & a_1 \\ a_1 & a_2 \end{vmatrix}$$

$$G = a_0^2a_3 - 3a_0a_1a_2 + 2a_1^3 = -\begin{vmatrix} 0 & a_0 & a_1 \\ a_0 & a_1 & a_2 \\ 2a_1 & a_2 & a_3 \end{vmatrix}$$